The number of elements in the periodic table

(a) Give a physical explanation why the *periodic table* has a finite number of elements, i.e. why atomic *nuclei* above a certain atomic number are unstable.

(b) A nucleus consists of protons and neutrons (nucleons) bound together by the *strong force*. Due to the "hard-core" close-range repulsion between nucleons, the nucleus can be approximated as a sphere of a constant density of nucleons. This is the *liquid drop model*. Suppose the nucleus contains $A = Z + N$ nucleons, where Z is the atomic number (number of protons), and N is the number of neutrons. Taking the size of a single nucleon to be $r_0 \approx 1\,\mathrm{fm}$, estimate the size of the entire nucleus.

(c) One of the simplest nuclei is the *deuteron*, which is a bound state of a proton and a neutron. The size of the deuteron is approximately $r_0 \approx 1\,\mathrm{fm}$. Use the uncertainty principle to estimate the nucleon binding energy V_s related to the strength of the strong force between nucleons.

(d) Using your answers to parts (b) and (c), estimate the atomic number of the heaviest stable element. Assume that the number of protons and neutrons in the nucleus is approximately equal: $Z \approx N \approx A/2$.

Solution

(a) The nucleons inside a nucleus are bound together by the strong force, which is approximately the same between proton-proton, neutron-neutron, and proton-neutron (this approximation is known as *isotopic invariance*). Protons carry positive electric charge, while neutrons are electrically neutral. Progressing down the periodic table towards the heavier elements (i.e., adding protons), the positive charge of the nucleus increases. Due to Coulomb repulsion, it costs more and more energy to add additional protons. The nucleon binding energy, however, remains approximately the same. When the Coulomb repulsion cost of adding a proton to the nucleus becomes larger than the nucleon binding energy, the nucleus becomes unstable, and can end its life via *α-decay*, *β-decay*, or *spontaneous fission*.

(b) Since the total number of nucleons is A, and their density is constant, the size of the entire nucleus is approximately

$$\boxed{R = r_0 A^{1/3}.} \tag{0.1}$$

(c) The wavefunctions of the nucleons in a deuteron are spatially confined on the length scale r_0. Therefore, from the uncertainty principle, the average momentum of each nucleon is $p \approx \hbar/r_0$. They thus have a kinetic energy $E \approx p^2/m_N$, where $m_N \approx 1\,\mathrm{GeV}/c^2 \approx 2000 m_e$ is the nucleon mass, and m_e is electron mass. Assuming this kinetic energy is on the order of their interaction energy (see the three-dimensional case in Prob. **??**), we estimate the strength of the strong interaction between nucleons:

$$\boxed{V_s \approx \frac{\hbar^2}{m_N r_0^2} = \frac{\hbar^2}{m_e a_B^2} \frac{m_e}{m_N} \frac{a_B^2}{r_0^2} \approx 30\,\mathrm{MeV}.} \tag{0.2}$$

(d) Suppose we have a nucleus of charge $Z = A/2$, and radius $R = r_0 A^{1/3} = r_0 (2Z)^{1/3}$. In order to add a proton, we have to overcome Coulomb repulsion

$$V_c = \frac{Ze^2}{R} \approx Z^{2/3} \frac{e^2}{a_B} \frac{a_B}{r_0} \approx 1\,\mathrm{MeV} \times Z^{2/3}. \tag{0.3}$$

This Coulomb repulsion is equal to the nucleon binding energy for

$$Z_{max} \approx (30\,\mathrm{MeV}/1\,\mathrm{MeV})^{3/2} \approx 160\,. \tag{0.4}$$

This is our estimate of the number of stable elements in nature. Nuclei with larger Z are unstable due to Coulomb repulsion between protons.

A more accurate calculation, considering fission of the nucleus into smaller nuclei, gives

$$\boxed{Z_{max} \approx 100\,.} \tag{0.5}$$