Bubble physics

(a) A spherical bubble is uniformly moving in a liquid. Describe the motion of the liquid in terms of the *velocity vector field*, i.e., in terms of the magnitude and direction of the liquid flow outside the bubble. Assume that the liquid is *incompressible* and *inviscid* (i.e., there is no friction), that the flow is *irrotational* (zero curl of velocity), and neglect the influence of the container walls.

(b) Based on the result of part (a), calculate the total kinetic energy of the liquid. Neglecting the mass of the gas in the bubble, what is the *effective mass* of the bubble moving in the liquid?

(c) What is the initial acceleration of a bubble released at rest (as in a glass of champagne that you will enjoy celebrating your Ph.D.)?

Solution

(a) We can reasonably assume that the bubble only perturbs the liquid in its vicinity, and that the fluid remains at rest far away from the bubble. Further, since the fluid is incompressible, the total flux into any volume within the fluid must be zero. This means that the divergence of the fluid velocity $\mathbf{v}$ must vanish:

$$\boldsymbol{\nabla} \cdot \mathbf{v} = 0. \tag{0.1}$$

In addition, we assume that, consistent with our neglecting viscous friction, the curl of the velocity field vanishes as well:

$$\boldsymbol{\nabla} \times \mathbf{v} = 0. \tag{0.2}$$

Now, this should immediately “ring a bell:” we have the same equations for the fluid velocity as we have for static electric and magnetic fields in free space. Therefore, solutions should also be similar.

The one last condition we need in order to determine the velocity field is that the fluid flows around the bubble, i.e., the component of the relative velocity of the fluid with respect to the bubble normal to the bubble's surface is zero.

In electromagnetism, we are used to solutions of problems involving, for example, a *dielectric polarizable sphere* in a uniform external field, in which the field produced by the sphere outside its borders is a *dipole field* (cf. also Prob. **??** with magnetic field). In the present case, a direct calculation immediately verifies that the following (educated) guess at the fluid-velocity distribution, indeed, satisfies all our requirements:

$$\mathbf{v}(\mathbf{r}) = \frac{1}{2}\frac{-\mathbf{v}_0 + 3(\mathbf{v}_0 \cdot \hat{\mathbf{r}})\hat{\mathbf{r}}}{(r/R)^3}, \tag{0.3}$$

where $\mathbf{v}_0$ is the bubble's velocity, $\hat{\mathbf{r}}$ is the unit vector along the direction of $\mathbf{r}$, which is the vector from the center of the bubble to the element of the fluid, and R is the bubble's radius.

Notice that the velocities at the “top” and at the “bottom” of the bubble (i.e., at the bubble's “poles”) are both equal to $\mathbf{v}_0$, so the fluid at these points, not too surprisingly, moves together with the bubble. In the equatorial plane, the fluid moves in the opposite direction of that of the bubble and with one half the speed (with respect to the bubble, the fluid speed is $3\mathrm{v}_0/2$).

Let us comment that nowhere in this solution we used the fact that the spherical object moving in the fluid is actually a bubble. Therefore, within the adopted approximations (such as the absence of *viscosity*), Eq. (0.3) holds also for a solid sphere or a sphere composed of another fluid.

(b) The total kinetic energy of the fluid is

$$K = 2\pi \int_0^\pi d\theta \int_R^\infty r^2 dr \frac{\rho \mathbf{v}(r,\theta)^2}{2}, \tag{0.4}$$

where θ is the polar angle between the bubble's axis and the fluid element, ρ is the fluid density, and the velocity $\mathbf{v}(r,\theta)$ is given by Eq. (0.3).

The integration is straightforward and yields for the total kinetic energy:

$$K = \frac{1}{2} \cdot \frac{4}{3}\pi R^3 \cdot \frac{\rho v_0^2}{2}. \tag{0.5}$$

Let us stress that the kinetic energy associated with the bubble's motion in the fluid, in fact, comes from the motion of the fluid, and not the gas in the bubble (whose contribution to the kinetic energy we neglect), and the effective mass of the bubble is one half the mass of the displaced fluid. This is an example of a common situation in *condensed-matter physics*, where the *effective mass* of an object, for example, an electron or a hole in a semiconductor, could be rather different from its "bare" mass.

(c) The *Archimedes' force* acting on the bubble is directed vertically, and its magnitude is equal to the weight of the displaced fluid. Since the effective mass of the bubble is one half the mass of the displaced fluid, the initial acceleration is $\boxed{2g}$, where $g \approx 9.8\ \mathrm{m/s^2}$, unless you are drinking your champagne on the moon.